Functions, Pt. 1

Review:
Proof techniques summary table.

	To ***prove*** that this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Assume A is true, then prove B is true.
$A \land B$	Prove A. Then prove B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

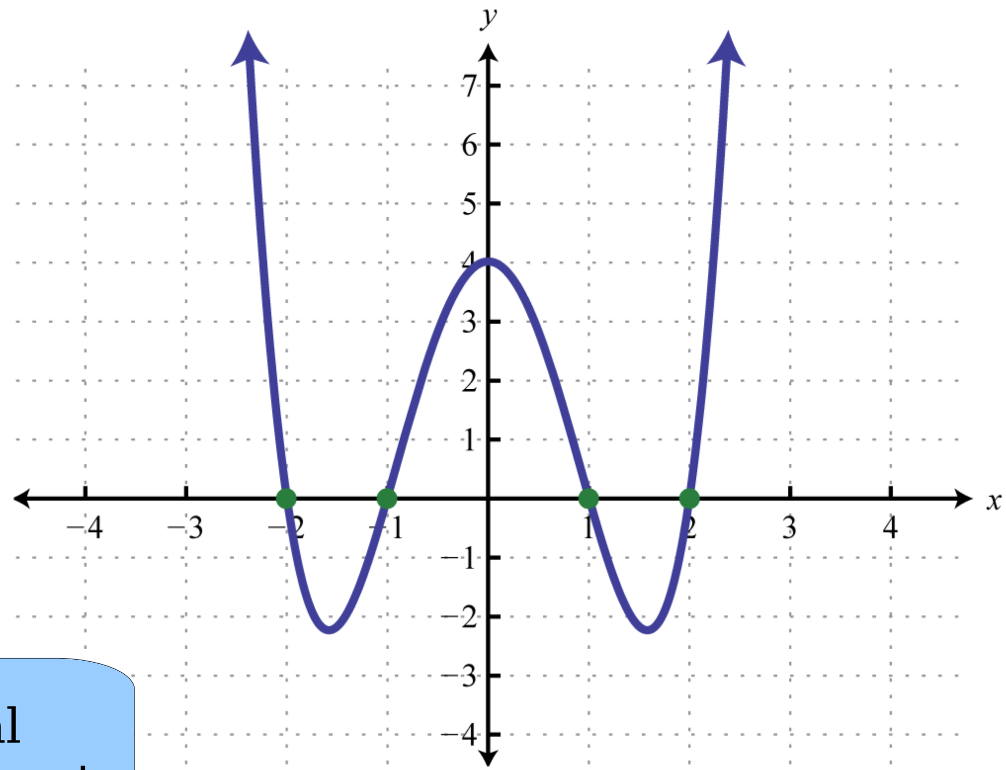
We'll refer to this several times today to help us write proofs. You'll find that although you've never written proofs about **Functions** before, it's just the same bag of tricks that we're used to!

Outline for Today

- ***What is a Function?***
  - It’s more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you’ll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

What is a function?

In high school math:



Take a real number as input

$$f(x) = x^4 - 5x^2 + 4$$

Give a real number as output

source: https://saylordotorg.github.io/text_intermediate-algebra/section_07/6aaf3a5ab540885474d58855068b64ce.png

In C++ coding:

```
int flipUntil(int n) {
  int numHeads = 0;
  int numTries = 0;

  while (numHeads < n) {
    if (randomBoolean()) {
      numHeads++;
    }
    numTries++;
  }

  return numTries;
}
```

Take input(s) of different type(s)

Return an output of some type

In logic, functions are ***deterministic***.

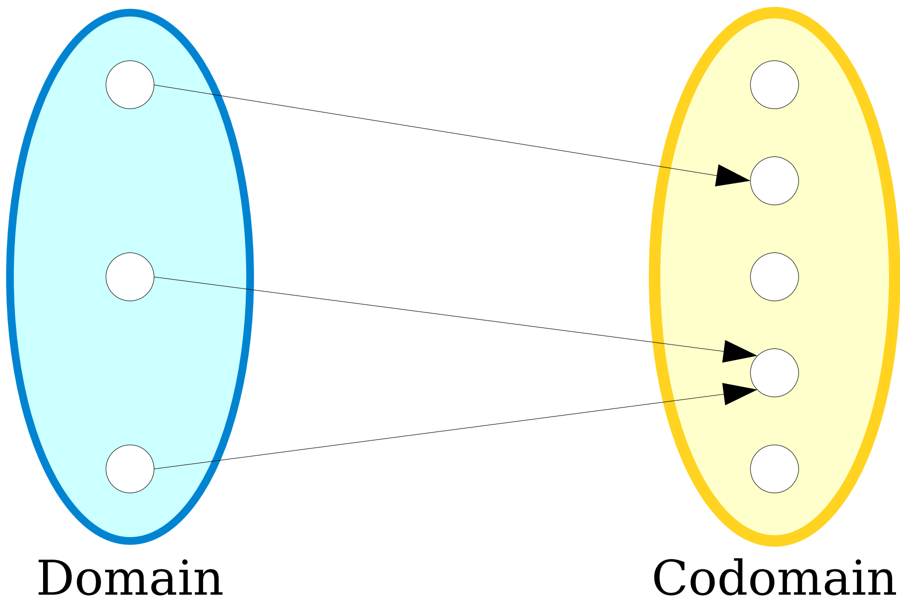
That is, given the same input, a function must always produce the same output.

In C++ code, we can use random numbers, but that would not be a valid function under our definition.

Domains and Codomains

- Every function *f* has two **sets** associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For all *x* in the domain, *f*(*x*) belongs to the codomain.

The function must be defined for every element of the domain.



The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.

Domains and Codomains

- Every function *f* has two **sets** associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For all *x* in the domain, *f*(*x*) belongs to the codomain.

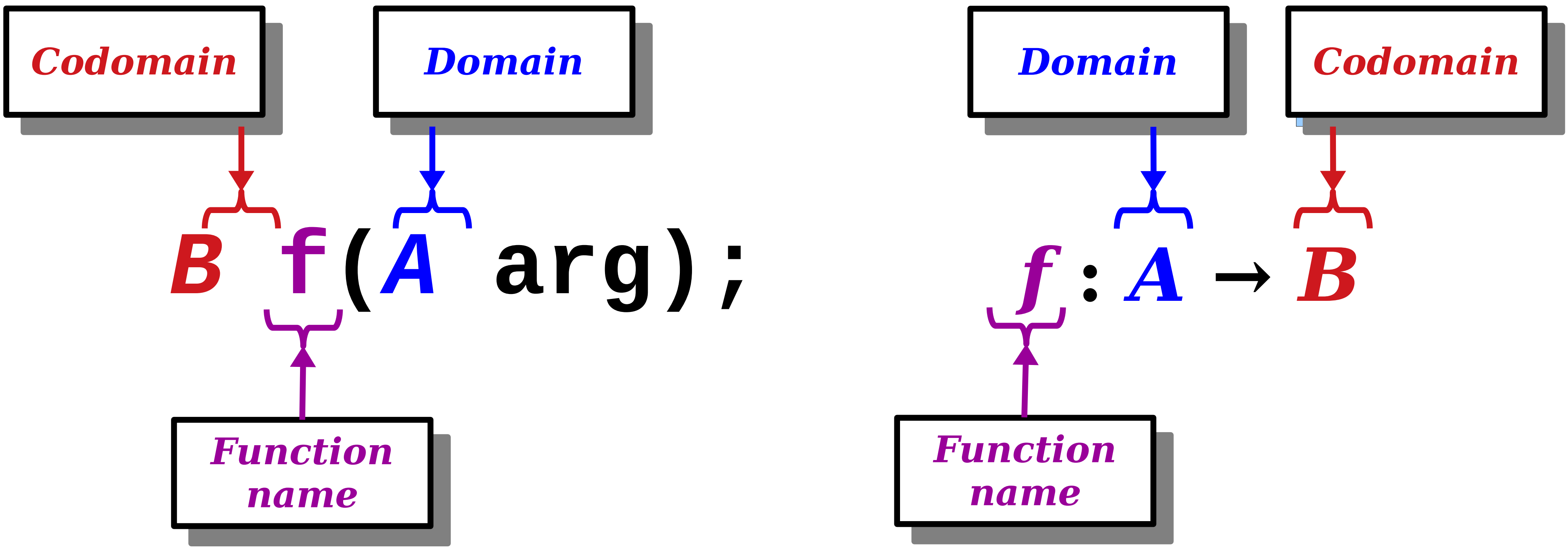
The ***domain*** of this function is ℝ. Any real number can be provided as input.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- Think of this like a “function prototype” in C++.



The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

If you're ever curious about whether something is a valid function, look back at these rules to decide. **The formal definition holds the answers!**

The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

PollEv.com/cs103spr26

The Official Rules for … s

- Formally speaking, we say that … following two rules hold.
- First, *f* must be obey its domain…

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, *f* must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

Can a function have an empty domain?

Defining Functions

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.

- All three pieces are necessary.
- There are a few ways to do this. Let’s go over a few examples.

Defining Functions

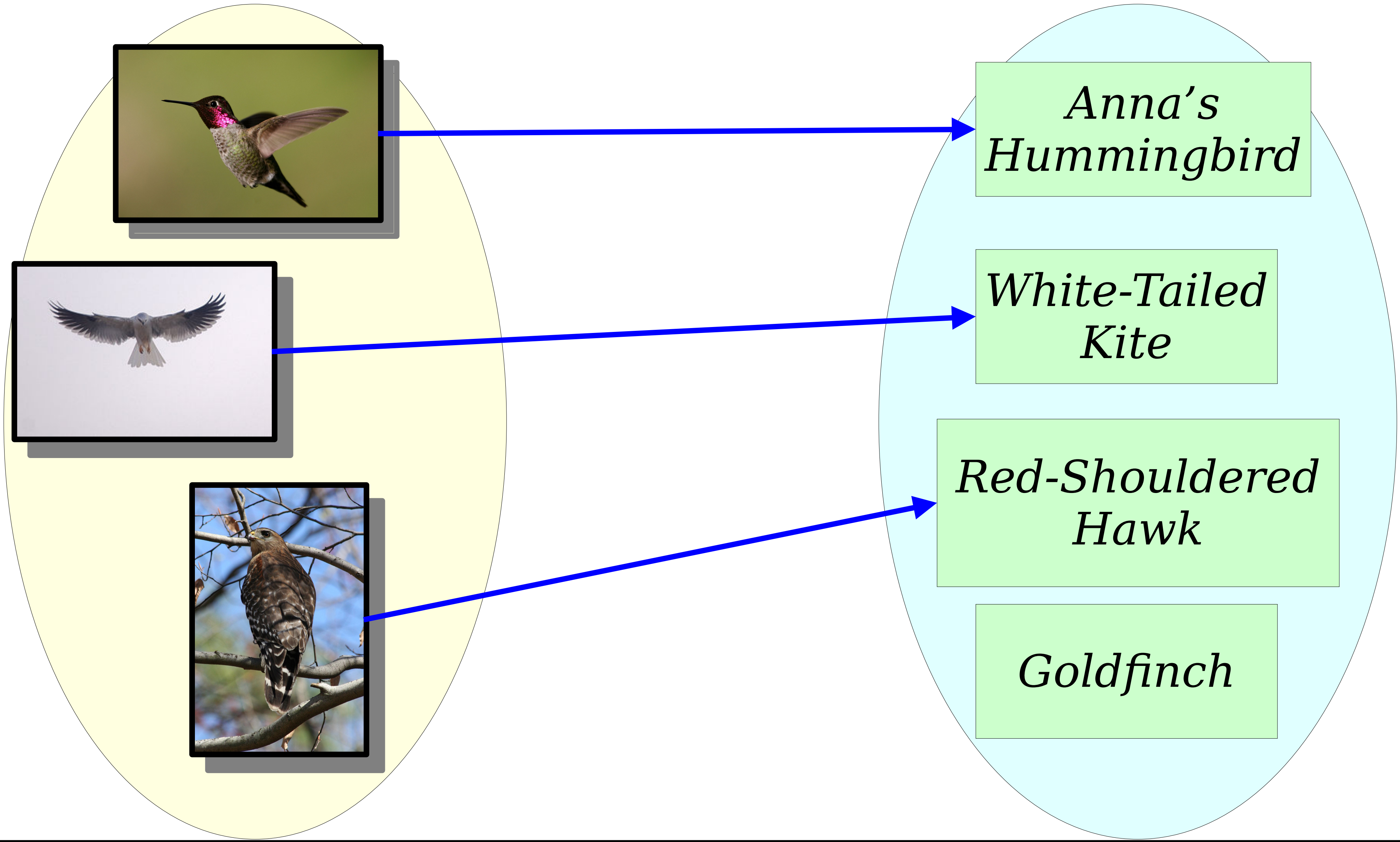
- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.

Usually these are handled by using the $f : A \to B$ notation.

Als
do
ab

- All three pieces are necessary.
- There are a few ways to do this. Let's go over a few examples.

Functions can be defined as a ***picture***.



Draw sets (ovals) to give the domain and codomain.
Draw a mapping (arrows) to define the function's action.

Functions can be defined as a ***rule***.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

Use the : notation to name the domain and codomain.
Use the $f(x)$ = notation to define the function's action.

Some rules are given ***piecewise***.

$$f : \mathbb{Z} \to \mathbb{N}, \text{ where}$$

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Again, both parts of the rule (: and $f(x)$) are necessary. Make sure at least one condition applies to each element of the domain, and that if more than one condition applies to the same element, they give the same answer!)

Some Nuances

$$f(x) = \frac{x+2}{x+1}$$

Quick Check:

If introduced as $f : \mathbb{N} \rightarrow \mathbb{R}$, would this be a valid function?

$$f(x) = \frac{x+2}{x+1}$$

Quick Check:

If introduced as $f : \mathbb{N} \to \mathbb{R}$, would this be a valid function?

Yep, it's a function! Every natural number maps to some real number.

$$f(x) = \frac{x+2}{x+1}$$

Quick Check:

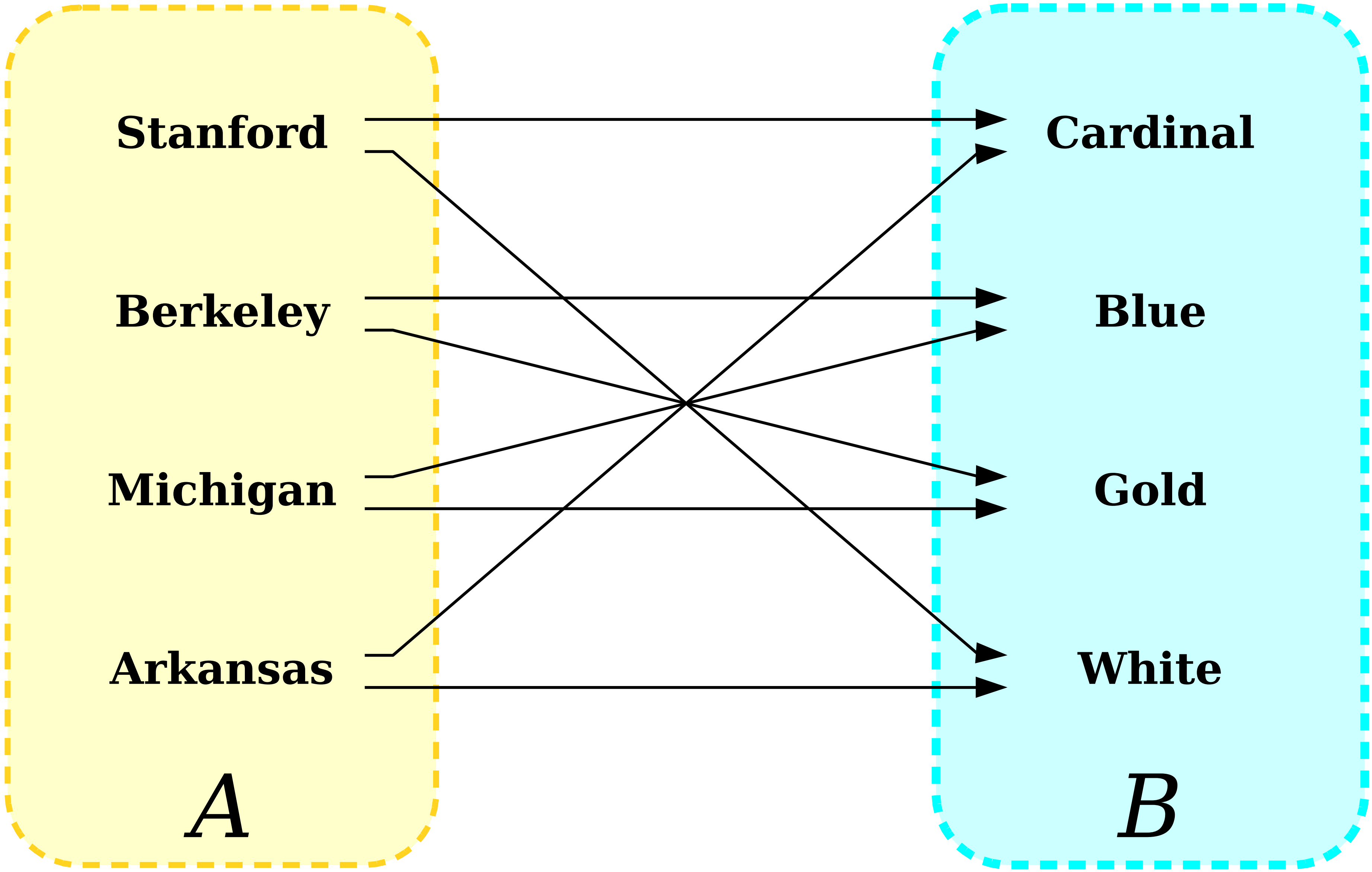
If introduced as $f : \mathbb{R} \to \mathbb{R}$, would this be a valid function?

$$f(x) = \frac{x+2}{x+1}$$

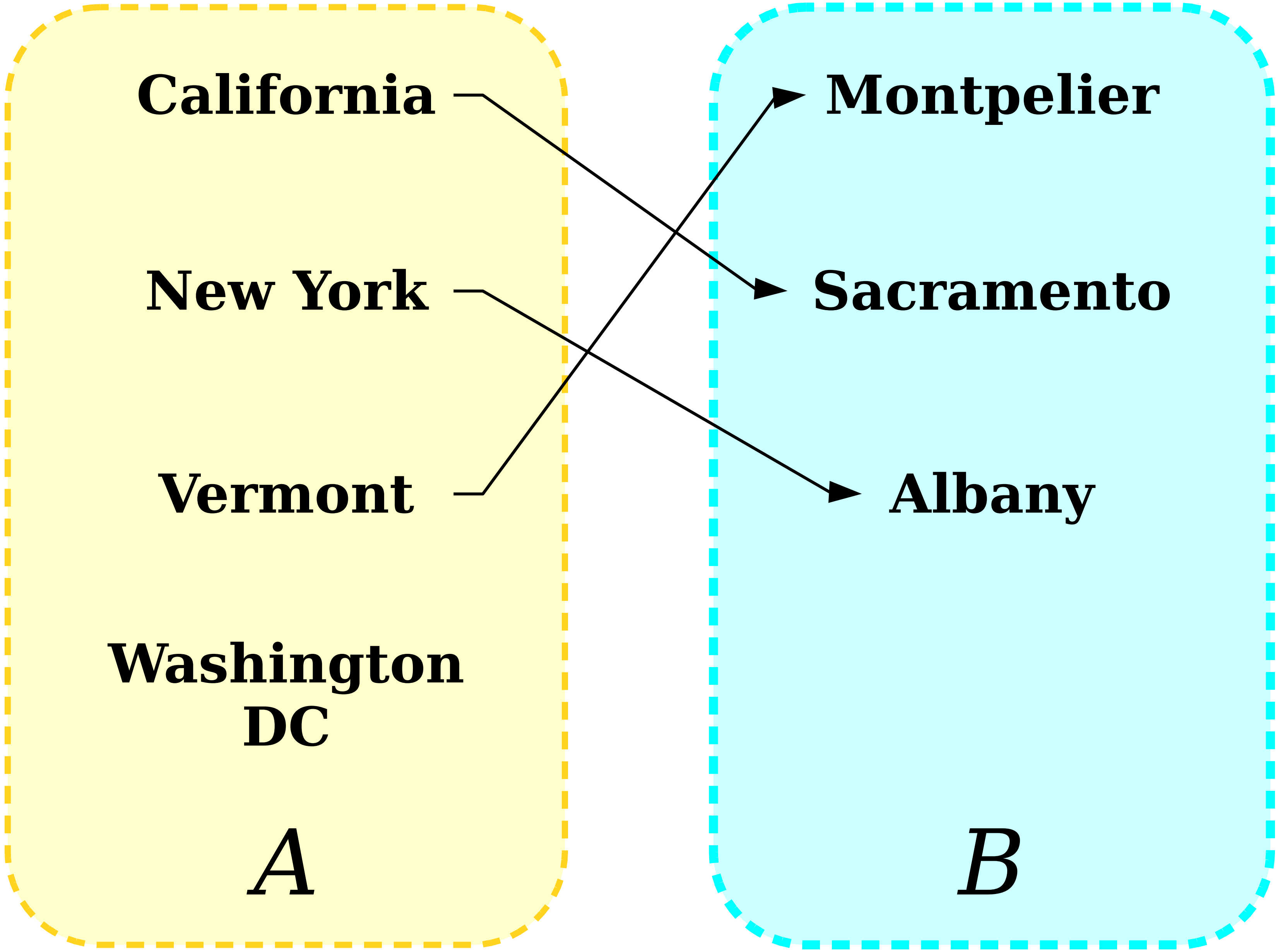
Quick Check:

If introduced as $f : \mathbb{R} \to \mathbb{R}$, would this be a valid function?

This expression isn't defined when x = -1, so f isn't defined over its full domain. We therefore don't consider it to be a function.

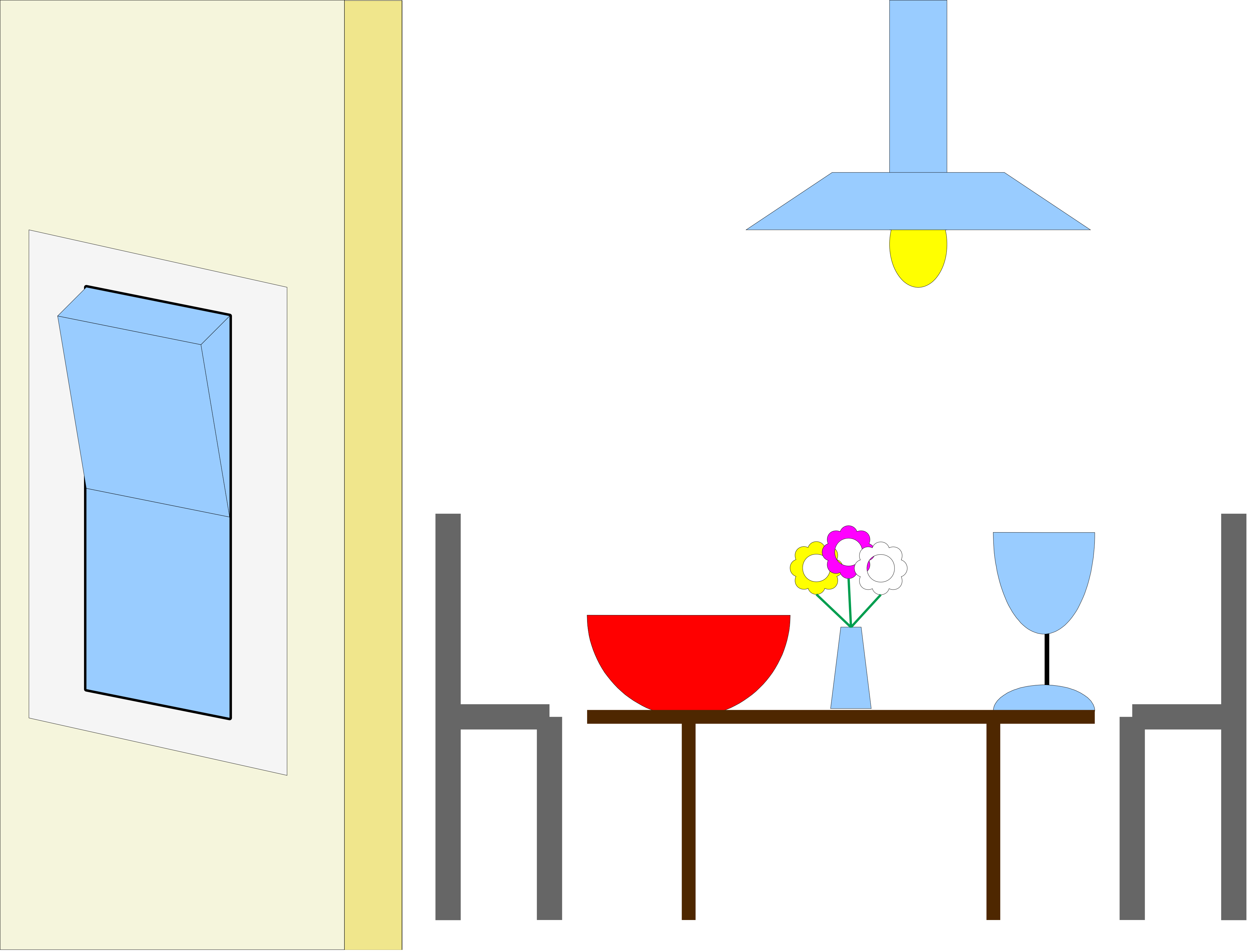


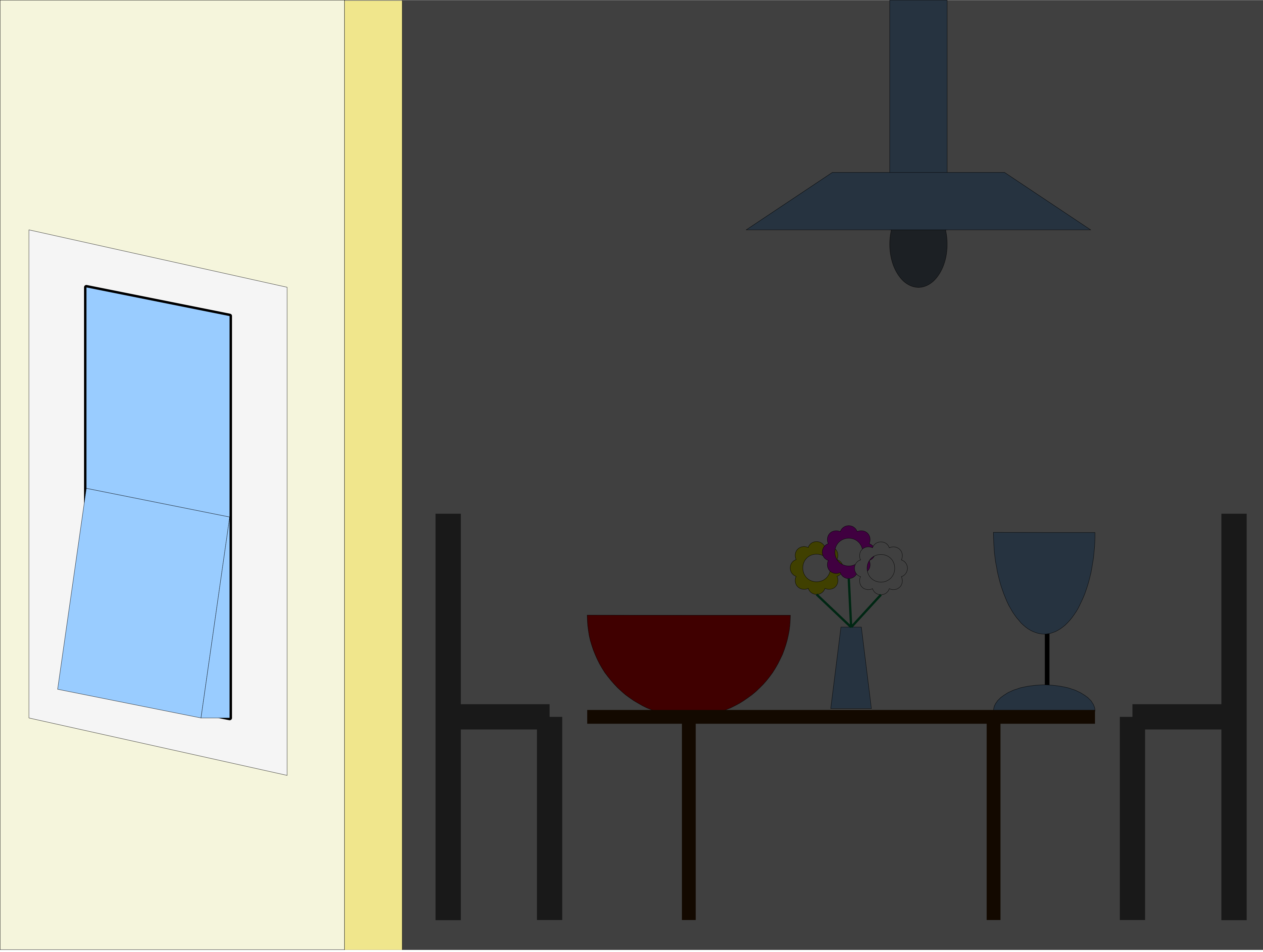
Is this a function from A to B?

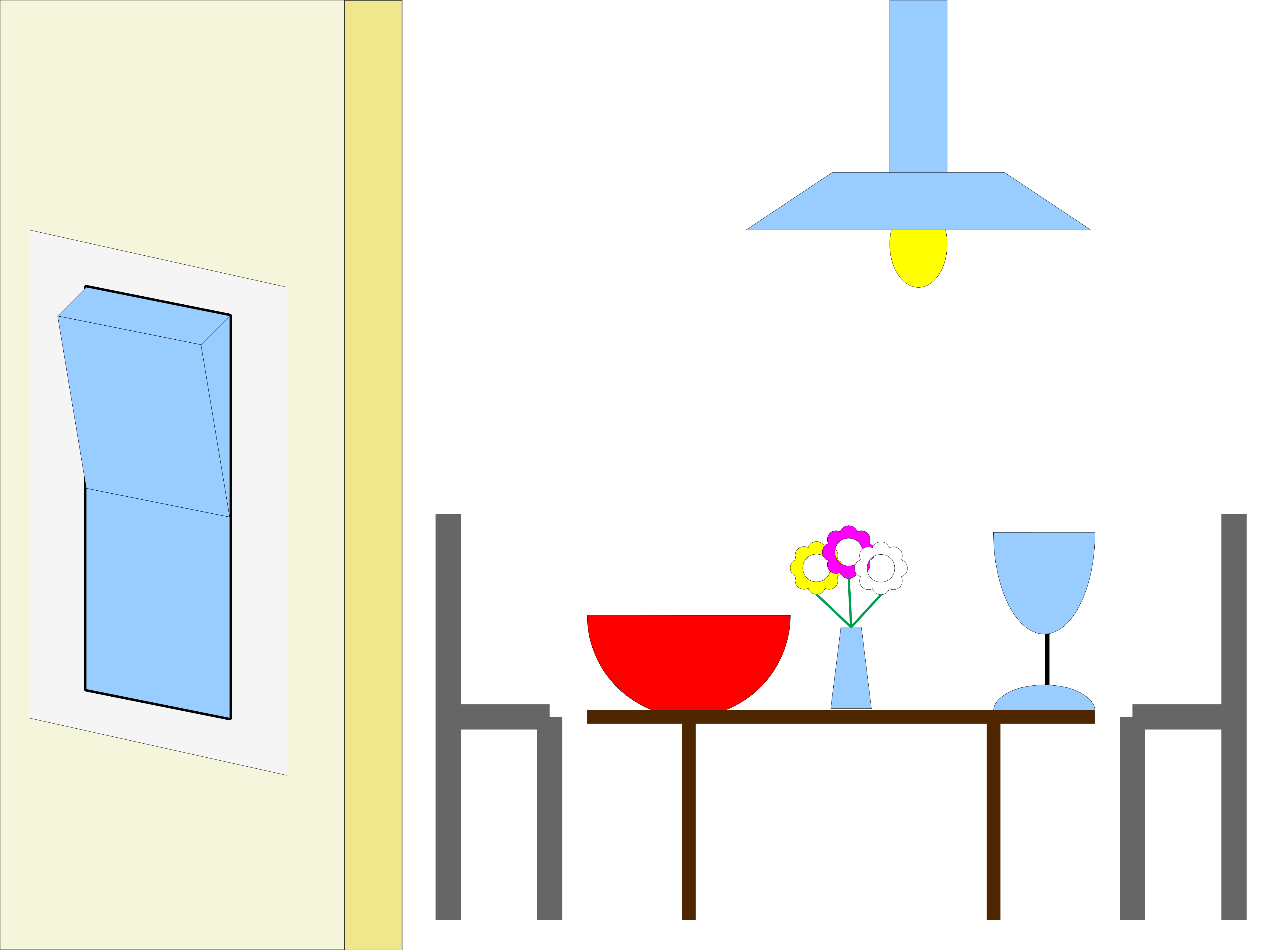


Is this a function from A to B?

Special Types of Functions







Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Theoretically unbreakable encryption (one-time pads).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

A function $f : A \to A$ *(notice this requires the domain and codomain to be the same set)* is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\boldsymbol{\forall x \in A.\ f(f(x)) = x.}$$

Involutions

PollEv.com/cs103spr26

- Which of the following are i
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

Select the number of involutions (0-4) on PollEv.

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^{1}/_{x}$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 1/x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$.
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$. *Not a function!*
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$. *Not a function!*
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$. *Not a function!*
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows: *Yep!*

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = {}^1/_x$. *Not a function!*
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows: *Yep!*

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions

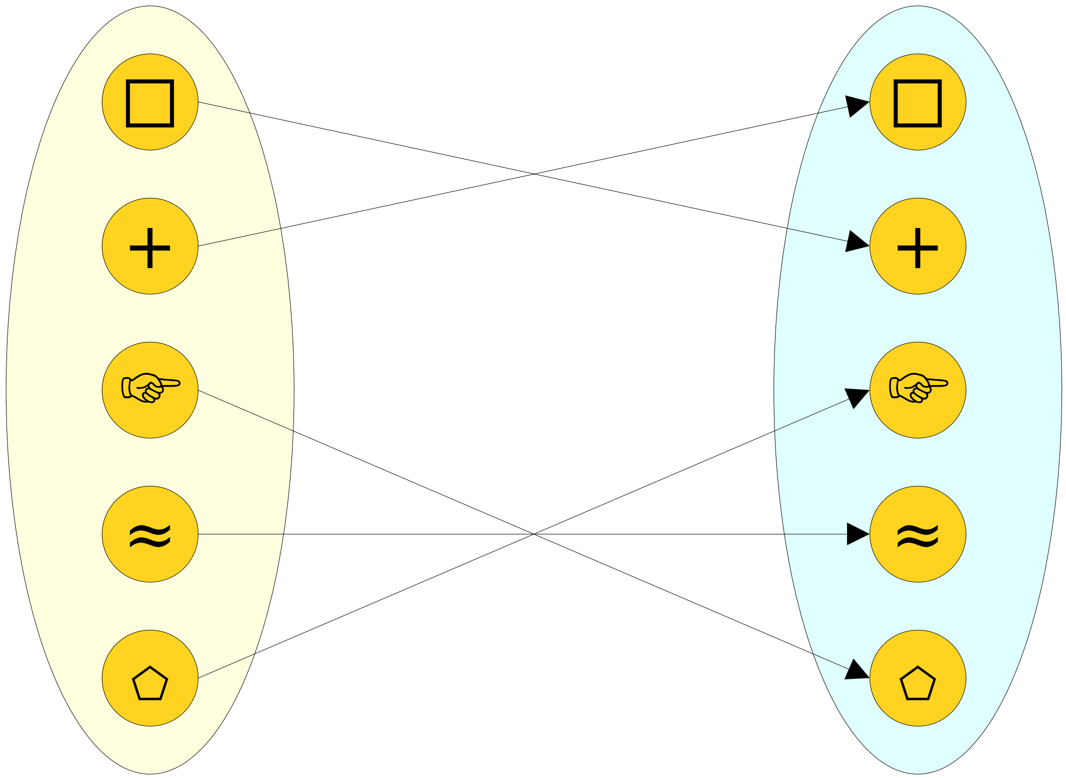
- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = -x$. *Yep!*
  - $f : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined as $f(x) = {}^1/_x$. ***Yep!***
  - $f : \mathbb{N} \to \mathbb{N}$ defined as follows: *Yep!*

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

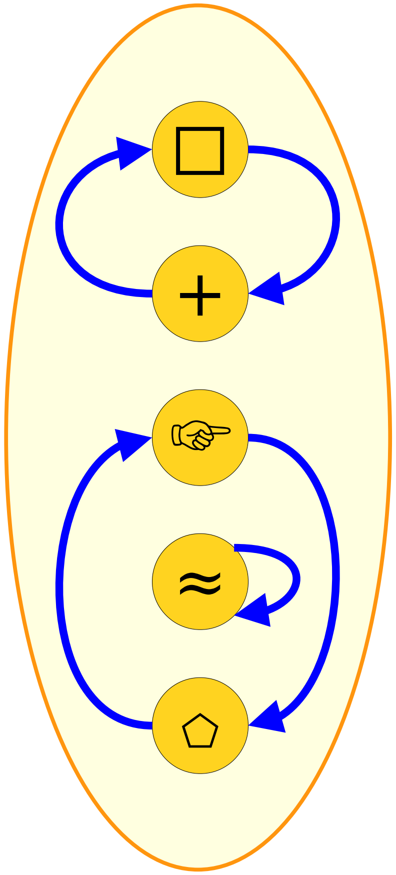
Involutions, Visually



A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Involutions, Visually



A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

For this problem, we will rely on two facts we will call Lemma 1 and Lemma 2 *("Lemma" just means theorem, but is used to label theorems we only use to help prove more important theorems, so like a "helper theorem")*, which you can assume are true, for this problem only:

Lemma 1: For all integers n, n is even if and only if $n + 1$ is odd.
Lemma 2: For all integers n, n is odd if and only if n - 1 is even.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

$$\forall n \in \mathbb{Z}.\ f(f(n)) = n.$$

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

$$\forall n \in \mathbb{Z}.\ f(f(n)) = n.$$

Therefore, we'll have the reader pick some $n \in \mathbb{Z}$, then argue that $f(f(n)) = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

$$\forall n \in \mathbb{Z}.\ f(f(n)) = n.$$

Therefore, we'll have the reader pick some $n \in \mathbb{Z}$, then argue that $f(f(n)) = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

$$\forall n \in \mathbb{Z}.\ f(f(n)) = n.$$

Therefore, we'll have the reader pick some $n \in \mathbb{Z}$, then argue that $f(f(n)) = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof:

What does it mean for f to be an involution?

$$\forall n \in \mathbb{Z}.\ f(f(n)) = n.$$

Therefore, we'll have the reader pick some $n \in \mathbb{Z}$, then argue that $f(f(n)) = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even.

Case 2: n is odd.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd, by Lemma 1. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2:

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd, by Lemma 1. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2: n is odd.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd, by Lemma 1. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2: n is odd. Then $f(n) = n - 1$, which is even, by Lemma 2. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd, by Lemma 1. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2: n is odd. Then $f(n) = n - 1$, which is even, by Lemma 2. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd, by Lemma 1. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2: n is odd. Then $f(n) = n - 1$, which is even, by Lemma 2. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$.

In either case, we see that $f(f(n)) = n$, which is what we need to show. ■

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even. Then $f(n) = n+1$, which is odd, by Lemma 1. This means that $f(f(n)) = f(n+1) = (n+1) - 1 = n$.

Case 2: n is odd. Then $f(n) = n - 1$, which is even, by Lemma 2. Then we see that $f(f(n)) = f(n - 1) = (n - 1) + 1 = n$.

In either case, we see that $f(f(n)) = n$, which is what we need to show. ■

!! Important style rule !!

This proof contains <u>no first-order logic syntax</u> (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Assume A is true, then prove B is true.
$A \land B$	Prove A. Then prove B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row of this proof techniques table did we use for that proof?

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg \forall n \in \mathbb{N}.\ f(f(n)) = n$$
$$\exists n \in \mathbb{N}.\ \neg(f(f(n)) = n)$$
$$\exists n \in \mathbb{N}.\ f(f(n)) \neq n$$

Therefore, we need to pick some concrete choice of n such that $f(f(n)) \neq n$.

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg \forall n \in \mathbb{N}.\ f(f(n)) = n$$
$$\exists n \in \mathbb{N}.\ \neg(f(f(n)) = n)$$
$$\exists n \in \mathbb{N}.\ f(f(n)) \neq n$$

Therefore, we need to pick some concrete choice of n such that $f(f(n)) \neq n$.

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg \forall n \in \mathbb{N}.\ f(f(n)) = n$$
$$\exists n \in \mathbb{N}.\ \neg(f(f(n)) = n)$$
$$\exists n \in \mathbb{N}.\ f(f(n)) \neq n$$

Therefore, we need to pick some concrete choice of n such that $f(f(n)) \neq n$.

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg\forall n \in \mathbb{N}.\ f(f(n)) = n$$
$$\exists n \in \mathbb{N}.\ \neg(f(f(n)) = n)$$
$$\exists n \in \mathbb{N}.\ f(f(n)) \neq n$$

Therefore, we need to pick some concrete choice of n such that $f(f(n)) \neq n$.

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

Proof: We need to show that there is some $n \in \mathbb{N}$ where $f(f(n)) \neq n$.

Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that $f(f(n)) \neq 2$, as required. ■

Theorem: The function $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$ is not an involution.

Proof: We need to show that there is some $n \in \mathbb{N}$ where $f(f(n)) \neq n$.

Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that $f(f(n)) \neq 2$, as required. ■

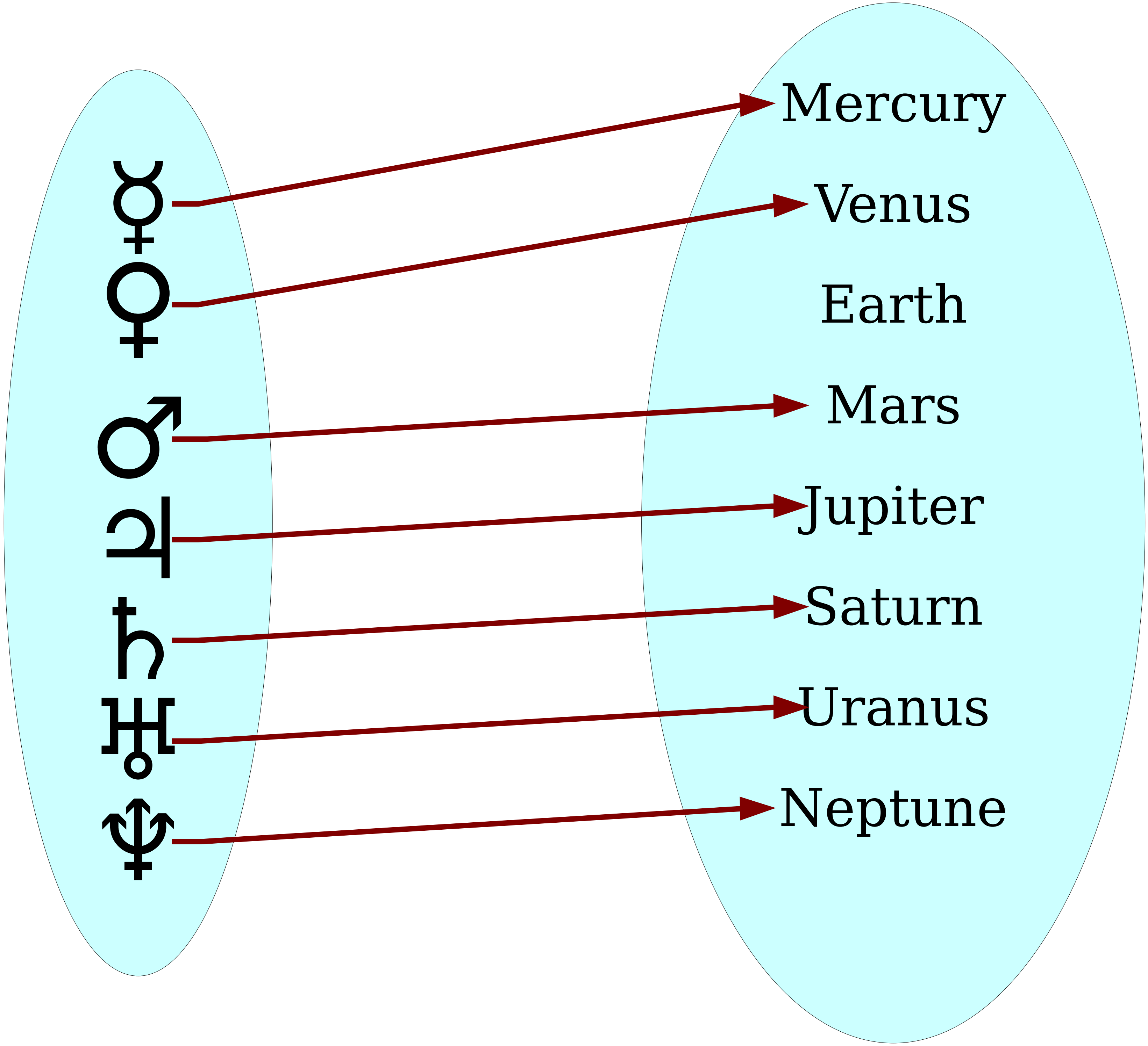
!! Important style rule !!

This proof contains <u>no first-order logic syntax</u> (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Assume A is true, then prove B is true.
$A \land B$	Prove A. Then prove B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row of this proof techniques table did we use for for that proof?

Another Class of Functions


☿
♀
♂
♃
♄
♅
♆
Mercury
Venus
Earth
Mars
Jupiter
Saturn
Uranus
Neptune

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injections

- Let 👬 be the set of all CS103 students. Which of the following are injective?
  - $f : $ 👬 $\to \mathbb{N}$ where $f(x)$ is x's Stanford ID number.
  - $f : $ 👬 $\to$ 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
  - $f : $ 👬 $\to$ 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

PollEv.com/
cs103spr26

Injections

- Let 🧑‍🤝‍🧑 be the set of all CS103
  Which of the following are inj
  - f : 🧑‍🤝‍🧑 → ℕ where $f(x)$ is x's Stanford ID number.
  - f : 🧑‍🤝‍🧑 → 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
  - f : 🧑‍🤝‍🧑 → 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

Select the number of injective functions (0-3) on PollEv.

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

Injections

- Let 🧑‍🤝‍🧑 be the set of all CS103 students. Which of the following are injective?

✔ - f : 🧑‍🤝‍🧑 → ℕ where $f(x)$ is x's Stanford ID number.
- f : 🧑‍🤝‍🧑 → 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
- f : 🧑‍🤝‍🧑 → 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

Injections

- Let 🧑‍🤝‍🧑 be the set of all CS103 students. Which of the following are injective?
  - ✔ f : 🧑‍🤝‍🧑 → ℕ where $f(x)$ is x's Stanford ID number.
  - ✘ f : 🧑‍🤝‍🧑 → 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
  - f : 🧑‍🤝‍🧑 → 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

Injections

- Let 🧑‍🤝‍🧑 be the set of all CS103 students. Which of the following are injective?
  - ✔ f : 🧑‍🤝‍🧑 → $\mathbb{N}$ where $f(x)$ is x's Stanford ID number.
  - ✘ f : 🧑‍🤝‍🧑 → 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
  - ✘ f : 🧑‍🤝‍🧑 → 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

Injections

- Let 👬 be the set of all CS103 students. Which of the following are injective?
  - ✔ f : 👬 → ℕ where $f(x)$ is x's Stanford ID number.
  - ✘ f : 👬 → 🌎, where 🌎 is the set of all countries and $f(x)$ is x's country of birth.
  - ✘ f : 👬 → 💬, where 💬 is the set of all given (first) names, where $f(x)$ is x's given (first) name.

A function $f : A \to B$ is ***injective*** if either statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$, assume $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

Therefore, we'll pick arbitrary n_1, $n_2 \in \mathbb{N}$, assume $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. (f(n_1) = f(n_2) \to n_1 = n_2)$$

$$\forall n_1 \in \mathbb{N}. \forall n_2 \in \mathbb{N}. (n_1 \neq n_2 \to f(n_1) \neq f(n_2))$$

Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$, assume $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof:

What does it mean for the function f to be injective?

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ f(n_1) = f(n_2) \to n_1 = n_2\ )$$

$$\forall n_1 \in \mathbb{N}.\ \forall n_2 \in \mathbb{N}.\ (\ n_1 \neq n_2 \to f(n_1) \neq f(n_2)\ )$$

Therefore, we'll pick arbitrary $n_1, n_2 \in \mathbb{N}$, assume $f(n_1) = f(n_2)$, then prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2$$

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required.

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

Good exercise: Repeat this proof using the other definition of injectivity!

Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

This in turn means that

$$2n_1 = 2n_2,$$

so $n_1 = n_2$, as required. ■

!! Important style rule !!

This proof contains <u>no first-order logic syntax</u> (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true...
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Assume A is true, then prove B is true.
$A \land B$	Prove A. Then prove B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row of this proof techniques table did we use for that proof?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \neg\forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ \neg(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land \neg(f(x_1) \neq f(x_2)))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \neg\forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ \neg(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land \neg(f(x_1) \neq f(x_2)))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \neg\forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ \neg(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land \neg(f(x_1) \neq f(x_2)))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \neg\forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ \neg(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land \neg(f(x_1) \neq f(x_2)))$$
$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$.

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1$$

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1,$$

so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required. ■

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

and

$$f(x_2) = f(1) = 1^4 = 1$$

so $f(x_1) = f(x_2)$ even though x

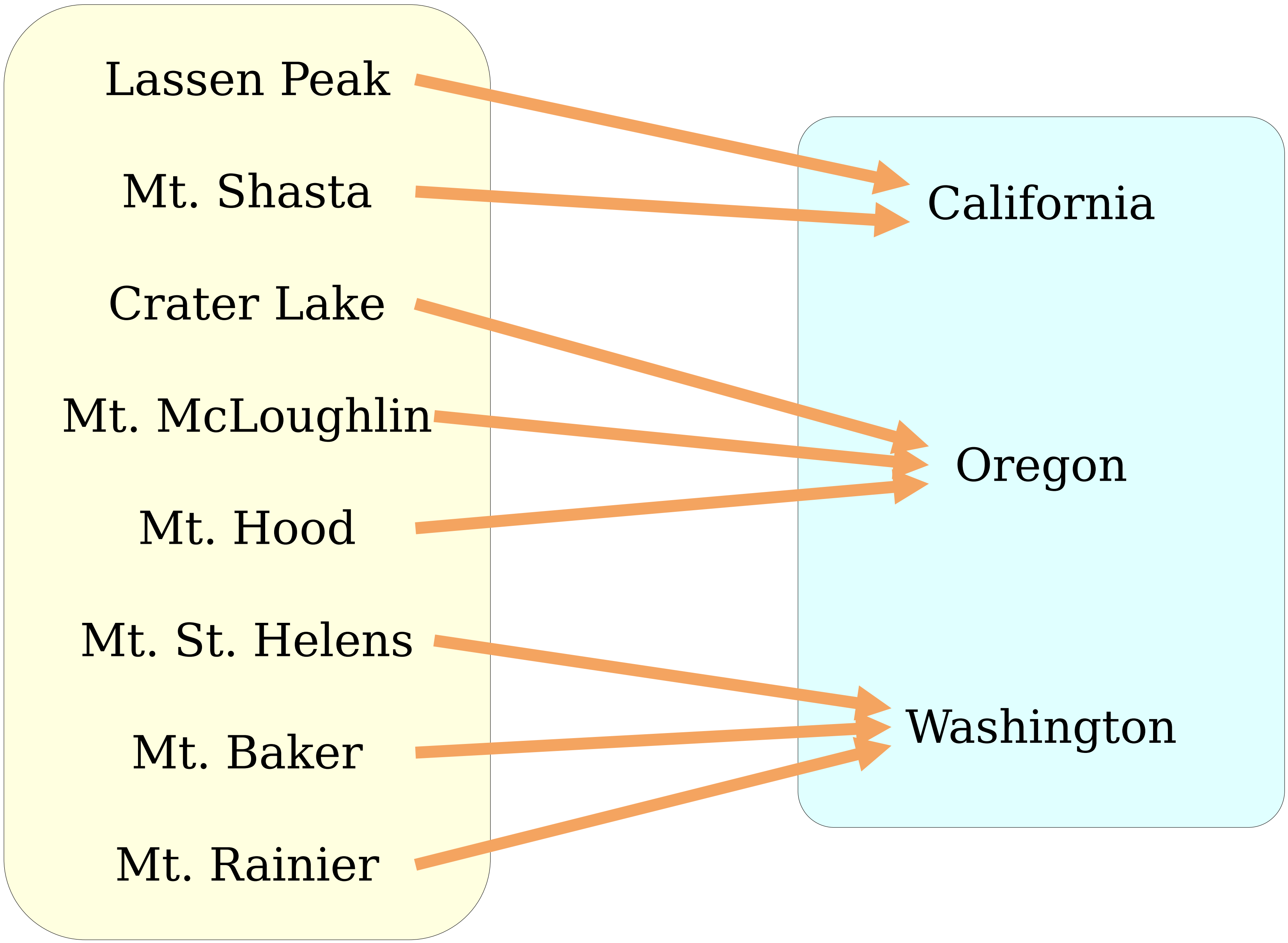
!! Important style rule !!

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Assume A is true, then prove B is true.
$A \land B$	Prove A. Then prove B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row of this proof techniques table did we use for that proof?

Another Class of Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof:

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof:

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2) = 2y / 2$$

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2) = 2y / 2 = y.$$

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2) = 2y / 2 = y.$$

So $f(x) = y$, as required.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2) = 2y / 2 = y.$$

So $f(x) = y$, as required. ■

!! Important style rule !!
This proof contains <u>no first-order logic syntax</u> (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$	Assume A is true, then prove B is true.
$A \land B$	Prove A. Then prove B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row of this proof techniques table did we use for that proof?

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg \forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \neg \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof:

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required. ■

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$. We need to show that $g(m) \neq n$.

Notice that $g(m) = 2m$ is even, while 137 is odd. Therefore, we have $g(m) \neq 137$, as required. ■

!! Important style rule !!
This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

	To ***prove*** that this is true…
$\forall x.\ A$	Have the reader pick an arbitrary *x*. We then prove *A* is true for that choice of *x*.
$\exists x.\ A$	Find an *x* where *A* is true. Then prove that *A* is true for that specific choice of *x*.
$A \to B$	Assume *A* is true, then prove *B* is true.
$A \land B$	Prove *A*. Then prove *B*.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.

Pop Quiz!
Which row of this proof techniques table did we use for that proof?

Recap from Today

- A ***function*** takes in an element of a ***domain*** and maps it to an element of a ***codomain***. Functions must be deterministic.
- Definitions are often given in first-order logic, and the structure of a first-order logic statement dictates the structure of a proof.
- ***Involutions*** and ***injections*** are specific classes of functions that have nice properties.

Next Time

- ***Surjections, Bijections***
  - Two new function types.
- ***Connecting Function Types***
  - Involutions, injections, surjections and bijections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.